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Three-dimensional Curve Motions Induced by the Modified Korteweg-de Vries Equation

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ABSTRACT

We have constructed one-phase quasi-periodic solutions of the curve equation induced by the mKdV equation. The solution is expressed in terms of the elliptic functions of Weierstrass. This solution can describe curve dynamics such as a vortex filament with axial velocity embedded in an incompressible inviscid fluid. There exist two types of curves (type-A, type-B) according to the form of the main spectra of the finite-band integrated solution. Our solution includes various filament shapes such as the Kelvin-type wave, the rigid vortex, plane curves, closed curves, and the Hasimoto one-solitonic filament.

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1 Introduction

Space curves are of interest in mathematics and physics. This is especially true with how these curves' mathematical structures are related to integrable equations. The first appearance of such a relationship was between the sine-Gordon equation and the differential geometry for surfaces of constant negative curvature [1]. Another interesting development was from Hasimoto who showed the motion of vortex filament motion is related to the nonlinear Schrödinger equation [2]. Motivated by this development, Lamb studied various helical curves that were described by well-known integrable equations [3]. He showed that there exists a curve that has a constant torsion and its curvature is described by the modified Korteweg-de Vries (mKdV) equation.

A special class zero torsion curves are the planar curves, which already appeared in the famous old problem of elastica. The elastica problem was solved by Euler. Typical solutions of the elastica problem, found by a numerical method, can be seen in the book [4], which are essentially those solutions of Euler. After that, this problem has occurred repeatedly in diverse forms. For example, Ishimori [5] found that the soliton solution of the mKdV equation is essentially the same as the loop soliton solution found in [6]. The loop soliton described the loop solitary wave propagating along a stretched rope. Mumford showed that computer vision problems were related to the mKdV equation [7]. A new form of the solution was given by Mumford which was expressed by the theta functions of the genus-one Riemann surface [7]. More general solutions, corresponding to the genus- N Riemann surfaces, were given by Matsutani in terms of the Weierstrass sigma functions and their generalizations [8]. On the other hand, [9] and [10] related the modified Korteweg-de Vries (mKdV) equation to motions of curves in a plane.

The above development of the elastica problem was confined to space curves having zero torsion and laying on a plane. On the other hand, the problem posed by Lamb, finding space curves related to the mKdV equation having a constant torsion and thus moving in three-dimensional space, was not investigated thoroughly in the research literature. As we can see in Section 2.4, this problem is related to the motion of a vortex filament with axial velocity and moving in an incompressible inviscid fluid [11]. Generally, constructing a curve having a given curvature and torsion is a difficult problem. It can be carried out for a simple form of curvature and torsion only [12]. For example, curves having a curvature of the one-soliton and constant non-zero torsion are explicitly constructed in the Appendix of [3].

Curves, having a more complex form of curvature, have been constructed for special cases only, by applying methods specially adopted to the given problem [13].

In our work, described in this paper, we construct space curves whose curvature is described by the quasi-periodic solution of the mKdV equation and curves which have constant non-zero torsion. The constructed curve is of the Kida-type stationary solution [14]. Equations that describe this Kida-type stationary solution were already known in [11], but were not solved because of their complicated form. The present calculation applied the method of the “modified finite-band integration”. The modified method was invented to solve the so-called effectivization problem, which is related to the extraction of the real solutions from the finite-band integration method [15, 16]. This modified method gives solutions in the simplest but important one-phase case. The modified method gave physical solutions in a rather simple form, which made this method useful to apply to real situations. Recently it was used to obtain quasi-periodic solutions of various integrable theories expressed by the Weierstrass elliptic functions [17, 18]. It was then applied to space curve problems, such as the Heisenberg model, filament motions under the localized induction approximation, and the Lund-Regge vortex motions [19, 20, 21]. These problems were related to the nonlinear Schrödinger equation and complex sine-Gordon equation, respectively.

A recent related calculation was the application of the Sym-Tafel formula to the problem of filament motions under the localized induction approximation [22]. Ref. [22] also obtained real quasi-periodic solutions expressed in terms of the Riemann theta function. This approach, based on the standard finite-band integration method, has the merit of obtaining more general solutions of multi-phase case. But these solutions through the higher-genus Riemann surfaces have complicated form, including numerical integrations to explicitly enumerate parameters appearing in the solutions. Thus it was difficult for physicists to apply the quasi-periodic solutions of these type to the analyses of experimental results. This was the reason of appearing one-phase solutions of various integrable equations in the previous paragraph by using the modified finite-band integration methods. Note that there generally exist multi-phase solutions of various integrable equations in terms of the Riemann theta function, though they are not widely used in the analyses of physical problems [17]. In fact, the present paper gives solutions expressed in terms of the Weierstrass’ elliptic functions with explicit parameters. Using the known properties of the Weierstrass’ functions, we can trace the variation of physical properties, like moving or rotation velocities, of the solutions along

with the spectral parameters.

The derivation of the curve configurations, described in the present paper, is similar to the previous cases of [19, 20, 21]. But there exists an important difference with the present calculation as compared to the previous ones, related with imposing conditions such as the constancy of torsion τ . It then results in a constraint on the main spectra and classifies the calculated curves into two categories (type-A and type-B). This restriction on the main spectra is not observed in the previous calculations of the modified finite-band integration method [17, 20, 21]. Solutions of a more simple type, discussed in [11], including the one-solitonic curve, the helicoidal vortex filament, and the Euler's elastica, are obtained by the reduction from the curve of the present calculation.

The general setup for the curve problem of the mKdV equation is given in Section 2. Explicit construction of the one-phase periodic solution is followed using a modified version of the finite-band integration method in Section 3. Especially, imposing special conditions of the mKdV problem is given in Section 3.4. The resulting formulae involve Weierstrass' elliptic functions. In Section 4, various specific solutions from the reduction of periodic solutions are explained using the functional relations of Weierstrass' functions. Some numerical plots, as well as an explicit check of solutions, are accomplished with the help of the symbolic package, Mathematica. Section 5 follows with a discussion.

2 mKdV-type space curves

2.1 Equations of curve motion

The space curve in three dimension is described by $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ vector (the unit tangent, normal, and binormal vectors, respectively). Here, $\mathbf{t} = \partial \mathbf{r} / \partial z$, where $\mathbf{r}(z, \bar{z})$ is the position vector of a point on the curve parametrized by the arc length z at time \bar{z} . They satisfy the following Serret-Frenet equation,

$$\partial \mathbf{t} = \kappa \mathbf{n}, \quad \partial \mathbf{n} = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \partial \mathbf{b} = -\tau \mathbf{n}, \quad (1)$$

where $\partial \equiv \partial / \partial z$ and κ, τ are respectively the curvature and torsion. Generally, motion of a point on the curve can be specified in the form

$$\dot{\mathbf{r}} \equiv \bar{\partial} \mathbf{r} = U \mathbf{n} + V \mathbf{b} + W \mathbf{t}, \quad (2)$$

where $\bar{\partial} \equiv \partial/\partial\bar{z}$ is the time derivative. Especially, the mKdV-type curves with a constant torsion τ is described by [3]

$$U = -\partial\kappa, \quad V = 2\kappa\tau, \quad W = 3\tau^2 - \kappa^2/2. \quad (3)$$

The family of curves with $\tau = 0$ are shown to be related with the problem of arc length conserving (nonstretching) motion and/or area conserving motion on a plane [9, 10]. It was first analyzed by Euler in his study of the thin rod, the so-called the elastica problem. Various possible shape of curves are illustrated in [4], including the famous figure-8 shape. These type of solutions are obtained by assuming that curves lye on a plane, thus leading to curves of zero torsion. On the contrary, the interesting problem of non-zero torsion has not been dealt in depth in literature and only the simple solutions has been obtained [3].

2.2 Integrable structure of the mKdV equation

We first start with the zero curvature condition of the mKdV equation,

$$[\partial + U_{KV}, \bar{\partial} + V_{KV}] = 0, \quad (4)$$

where

$$\begin{aligned} U_{KV} &= \begin{pmatrix} 0 & \kappa/2 \\ -\kappa/2 & 0 \end{pmatrix} + i\tau/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ V_{KV} &= \begin{pmatrix} 0 & -(\partial^2\kappa)/2 - \kappa^3/4 \\ (\partial^2\kappa)/2 + \kappa^3/4 & 0 \end{pmatrix} + i\tau \begin{pmatrix} -\kappa^2/4 & (\partial\kappa)/2 \\ (\partial\kappa)/2 & \kappa^2/4 \end{pmatrix} \\ &+ \tau^2 \begin{pmatrix} 0 & \kappa/2 \\ -\kappa/2 & 0 \end{pmatrix} + i\tau^3/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (5)$$

The mKdV variable $\kappa = \kappa(z, \bar{z})$ becomes the curvature of the corresponding curve problem, while the spectral parameter τ becomes the torsion. Note that τ is a constant and it does not depend on z, \bar{z} . The zero curvature condition in Eq. (4) is the compatibility condition for the overdetermined system of the following associated linear equations

$$(\partial + U_{KV})\Phi = 0, \quad (\bar{\partial} + V_{KV})\Phi = 0. \quad (6)$$

By explicitly enumerating the zero curvature condition at the level of $O(\tau^0)$, we obtain the mKdV equation,

$$\bar{\partial}\kappa + \partial^3\kappa + \frac{3}{2}\kappa^2\partial\kappa = 0, \quad (7)$$

while other levels of $O(\tau^i), i = 1, 4$ just becomes identities.

2.3 The Sym-Tafel formula

Now we introduce the duality relation between the curve problem and the mKdV system. We write down the curve position vector $\mathbf{r} \equiv (r_1, r_2, r_3)$ in terms of Φ in Eq. (6) and shows that the linear equation in Eq. (6) implies the Serret-Frenet equation in Eq. (1). For this, we introduce the following Sym-Tafel formula,

$$r = 2i\Phi^{-1}\frac{\partial}{\partial\tau}\Phi, \quad (8)$$

where $r \equiv \sum r_i \sigma_i, i = 1, 3$, with the Pauli matrices σ_i [23, 24]. (From now on, we will omit the summation notation.)

To check that Eq. (8) gives the Serret-Frenet equation, we first notice that

$$\partial r = -2i\Phi^{-1}\partial\Phi\Phi^{-1}\frac{\partial}{\partial\tau}\Phi + 2i\Phi^{-1}\frac{\partial}{\partial\tau}(\partial\Phi) = -2i\Phi^{-1}\frac{\partial U_{KV}}{\partial\tau}\Phi = \Phi^{-1}\sigma_3\Phi. \quad (9)$$

In other words, the unit tangent vector $\mathbf{t} = \partial\mathbf{r} \equiv (t_1, t_2, t_3)$ is given by

$$t = t_i \sigma_i = \partial r_i \sigma_i = \partial r = \Phi^{-1}\sigma_3\Phi. \quad (10)$$

Note that $t_i t_i = \frac{1}{2}\text{Tr}t^2 = 1$. This equation shows that the tangent vector \mathbf{t} is given by the rotation of \hat{k} (the unit vector along the z -axis), where the rotation is induced by the similarity transformation of Φ . In a similar way, we find that

$$\partial t = \partial t_i \sigma_i = \partial^2 r = -[\Phi^{-1}\partial\Phi, \Phi^{-1}\sigma_3\Phi] = \Phi^{-1}[U_{KV}, \sigma_3]\Phi = -\kappa\Phi^{-1}\sigma_1\Phi. \quad (11)$$

Eq. (11) is one of the Serret-Frenet equation, $\partial\mathbf{t} = \kappa\mathbf{n}$ (κ is the curvature), when we define the normal vector $\mathbf{n} \equiv (n_1, n_2, n_3)$ as

$$n = n_i \sigma_i = -\Phi^{-1}\sigma_1\Phi. \quad (12)$$

Thus, the normal vector \mathbf{n} is given by the rotation of \hat{i} (the unit vector along the x -axis). It is clear that \mathbf{t} and \mathbf{n} are orthogonal to each other. Now,

$$\partial n = \partial n_i \sigma_i = -\partial(\Phi^{-1}\sigma_1\Phi) = [\Phi^{-1}\partial\Phi, \Phi^{-1}\sigma_1\Phi] = -\kappa\Phi^{-1}\sigma_3\Phi + \tau\Phi^{-1}\sigma_2\Phi, \quad (13)$$

which gives another Serret-Frenet equation $\partial\mathbf{n} = \tau\mathbf{b} - \kappa\mathbf{t}$, when we take the binormal vector $\mathbf{b} \equiv (b_1, b_2, b_3)$ as

$$b \equiv b_i \sigma_i = \Phi^{-1}\sigma_2\Phi. \quad (14)$$

In this case, the binominal vector \mathbf{b} is given by the rotation of \hat{j} (the unit vector along the y -axis). Finally, similar calculation gives

$$\partial b = \partial b_i \sigma_i = \partial(\Phi^{-1} \sigma_2 \Phi) = -[\Phi^{-1} \partial \Phi, \Phi^{-1} \sigma_2 \Phi] = \tau \Phi^{-1} \sigma_1 \Phi, \quad (15)$$

which gives another Serret-Frenet equation, $\partial \mathbf{b} = -\tau \mathbf{n}$.

The time dependence of the r is similarly constructed as

$$\begin{aligned} \bar{\partial} r &= -2i\Phi^{-1} \bar{\partial} \Phi \Phi^{-1} \frac{\partial}{\partial \tau} \Phi + 2i\Phi^{-1} \frac{\partial}{\partial \tau} (\bar{\partial} \Phi) = -2i\Phi^{-1} \frac{\partial V_{KV}}{\partial \tau} \Phi \\ &= \Phi^{-1} (3\tau^2 \sigma_3 + 2\kappa \tau \sigma_2 - \frac{1}{2} \kappa^2 \sigma_3 + \partial \kappa \sigma_1) \Phi \\ &= 3\tau^2 t + 2\kappa \tau b - \frac{1}{2} \kappa^2 t - \partial \kappa n, \end{aligned} \quad (16)$$

where t and b are given by Eqs. (10), (12) and (14). It is just the Eq. (3) and is already found by Lamb in [3]. This equation can be rewritten in terms of the curve variable \mathbf{r} ,

$$\bar{\partial} \mathbf{r} = -\partial^3 \mathbf{r} + \left(\frac{3}{2} |\partial^2 \mathbf{r}|^2 - 3\tau^2 \right) \partial \mathbf{r} + 3\tau \partial^2 \mathbf{r} \times \partial \mathbf{r}. \quad (17)$$

2.4 Equation for localized induction approximation with axial velocity

Eq. (17) is the curve equation induced from the mKdV equation with the spectral parameter τ . This equation can be rewritten to the equation for localized induction approximation generalized to take account of the axial-flow effect. Consider the following change of coordinates,

$$z \rightarrow X - \frac{c^2}{3} T, \quad \bar{z} \rightarrow T, \quad \tau \rightarrow -\frac{c}{3} \quad (18)$$

Above coordinates change transforms Eq. (17) into the following form,

$$\partial_T \mathbf{r} = -\partial_X^3 \mathbf{r} - \frac{3}{2} |\partial_X \mathbf{r}|^2 \partial_X \mathbf{r} - c \partial_X^2 \mathbf{r} \times \partial_X \mathbf{r}, \quad (19)$$

where $\partial_T \equiv \frac{\partial}{\partial T}$, $\partial_X \equiv \frac{\partial}{\partial X}$. Eq. (19) is the equation introduced in [11] and it describes the motion of a thin vortex filament with axial velocity, embedded in a inviscid incompressible fluid. In [11], solutions of Eq. (19) were introduced including the N-soliton solution, a circular helix and a plane curve of Euler's elastica. But the Kida-type solution was not given explicitly and only equations describing the Kida-type curve was presented. Our results can be used to fill up this gap by using the transformation in Eq. (18).

2.5 Integrable structure of the curve equation

We now use a modified form of the R-transformation [25, 20, 21] to obtain the Lax pair of the curve equation. It will be used to obtain a quasi-periodic solution by applying the finite integration method in the next section. First we define $\Psi \equiv \Phi^{-1}\hat{\Phi}$ where $\Phi, \hat{\Phi}$ are solutions of the linear equation (6) with the spectral parameter $\tau, \hat{\tau}$, respectively. Define

$$\begin{aligned}
M &\equiv \partial\Psi\Psi^{-1} = -\Phi^{-1}\partial\Phi + \Phi^{-1}\partial\hat{\Phi}\hat{\Phi}^{-1}\Phi = \Phi^{-1}U_{KV}(\tau)\Phi - \Phi^{-1}U_{KV}(\hat{\tau})\Phi \\
&= \frac{i}{2}(\tau - \hat{\tau})\Phi^{-1}\sigma_3\Phi = \frac{i}{2}(\tau - \hat{\tau})\partial r, \\
N &\equiv \bar{\partial}\Psi\Psi^{-1} = \frac{i}{2}(\tau^3 - \hat{\tau}^3)\Phi^{-1}\sigma_3\Phi + \frac{1}{2}(\tau^2 - \hat{\tau}^2)\Phi^{-1}\begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix}\Phi \\
&\quad + \frac{i}{4}(\tau - \hat{\tau})\Phi^{-1}\begin{pmatrix} -\kappa^2 & 2\partial\kappa \\ 2\partial\kappa & \kappa^2 \end{pmatrix}\Phi \\
&= \frac{i}{2}(-2\tau^3 + 3\hat{\tau}\tau^2 - \hat{\tau}^3)\partial r + \frac{1}{4}(\tau^2 - 2\hat{\tau}\tau + \hat{\tau}^2)[\partial r, \partial^2 r] + \frac{i}{2}(\tau - \hat{\tau})\bar{\partial}r. \quad (20)
\end{aligned}$$

Note that $M\Psi = \partial\Psi$, $N\Psi = \bar{\partial}\Psi$. Now, we introduce $\lambda \equiv \hat{\tau} - \tau$, which becomes a new spectral parameter of the following linear equation for r ,

$$(\partial - M)\Psi = 0, \quad (\bar{\partial} - N)\Psi = 0, \quad (21)$$

where

$$\begin{aligned}
M &= -\frac{i}{2}\lambda\partial r, \\
N &= -\frac{i}{2}\lambda^3\partial r - i\frac{3}{2}\lambda^2\tau\partial r + \frac{1}{4}\lambda^2[\partial r, \partial^2 r] - \frac{i}{2}\lambda\bar{\partial}r. \quad (22)
\end{aligned}$$

Note that the compatibility condition of the associated linear equation, i.e., $[\partial - M, \bar{\partial} - N] = 0$ gives the following equation of motion for r ,

$$i\frac{3}{2}\tau\partial^2 r - \frac{1}{4}[\partial r, \partial^3 r] - \frac{1}{4}[\partial r, \bar{\partial}r] = 0, \quad (23)$$

as well as the identity $\text{Tr}(\partial r)^2 = 2$. Eq. (23) gives **n** and **b** components of $\bar{\partial}r$ in Eq. (2), which are $U = -\partial\kappa, V = 2\kappa\tau$ in Eq. (3). But Eq. (23) does not give the **t** components of $\bar{\partial}r$, i.e., $W = 3\tau^2 - \kappa^2/2$ in Eq. (3). This fact is not surprising since W is known not to be related with the intrinsic form of the space curve, and is related with the internal parametrization of the curve [10]. In this respect, Eq. (23) just describes the intrinsic structure of the curve. In the next section, we will derive the quasi-periodic solution of the mKdV-type curve, starting from the associated linear equation in Eq. (21). In the course of derivation, we will additionally require the condition $W = 3\tau^2 - \kappa^2/2$, as well as the constancy of τ , see section 3.4.

3 One-phase quasi-periodic solution

3.1 Squared wavefunctions

First we introduce notations,

$$\partial r = \begin{pmatrix} \alpha_3 & \alpha_+ \\ \alpha_- & -\alpha_3 \end{pmatrix}, \quad \bar{\partial} r = \begin{pmatrix} \beta_3 & \beta_+ \\ \beta_- & -\beta_3 \end{pmatrix}, \quad [\partial r, \partial^2 r] = \begin{pmatrix} \gamma_3 & \gamma_+ \\ \gamma_- & -\gamma_3 \end{pmatrix}, \quad (24)$$

where $\alpha_+^* = \alpha_-$, $\alpha_3^* = \alpha_3$, $\beta_+^* = \beta_-$, $\beta_3^* = \beta_3$, $\gamma_+^* = -\gamma_-$, $\gamma_3^* = -\gamma_3$. The equation of motion in Eq. (23) is rewritten as

$$\begin{aligned} 6i\tau\partial\alpha_3 - \partial\gamma_3 &= \alpha_+\beta_- - \alpha_-\beta_+, \\ 3i\tau\partial\alpha_+ - \frac{1}{2}\partial\gamma_+ &= \alpha_3\beta_+ - \alpha_+\beta_3, \end{aligned} \quad (25)$$

while the constraint $\text{Tr}(\partial r)^2 = 2$ becomes $\alpha_3^2 + \alpha_+\alpha_- = 1$.

The finite-band integration method relies on the fact that the curve satisfying Eq. (23) guarantees the existence of the solution Ψ of Eq. (21) as a function of λ . The following procedure assumes that the squared function of Ψ is a polynomial function in λ , and thus is called by the “modified squared wavefunctions method” [16]. Let the systems (21) have two basic solutions of column matrices, $\Psi_1 = (\psi_1, \psi_2)$ and $\Psi_2 = (\phi_1, \phi_2)$, which is used to build a squared wave function such that

$$f = -(i/2)(\psi_1\phi_2 + \psi_2\phi_1), \quad g = \psi_1\phi_1, \quad h = -\psi_2\phi_2. \quad (26)$$

From the definition of f, g and h in Eq. (26) and using Eqs. (21), (22) and (24), we can obtain following equations,

$$\begin{aligned} \partial f &= -\frac{\lambda}{2}\alpha_-g + \frac{\lambda}{2}\alpha_+h, \quad \partial g = \lambda\alpha_+f - i\lambda\alpha_3g, \\ \bar{\partial} f &= \left(\frac{1}{2}\lambda^3\alpha_+ + \frac{3}{2}\lambda^2\tau\alpha_+ + \frac{i}{4}\lambda^2\gamma_+ + \frac{1}{2}\lambda\beta_+\right)h - \left(\frac{1}{2}\lambda^3\alpha_- + \frac{3}{2}\lambda^2\tau\alpha_- + \frac{i}{4}\lambda^2\gamma_- + \frac{1}{2}\lambda\beta_-\right)g, \\ \bar{\partial} g &= \left(-i\lambda^3\alpha_3 - 3i\lambda^2\tau\alpha_3 + \frac{1}{2}\lambda^2\gamma_3 - i\lambda\beta_3\right)g + \left(\lambda^3\alpha_+ + 3\lambda^2\tau\alpha_+ + \frac{i}{2}\lambda^2\gamma_+ + \lambda\beta_+\right)f. \end{aligned} \quad (27)$$

Equations for h are similarly given.

Using Eqs. (27), it can be explicitly checked that $P(\lambda) \equiv f^2 - gh$ is independent of z and \bar{z} and is only the function of λ . The so-called 1-phase periodic solution is obtained by specially taking the form of

$$P(\lambda) = f^2 - gh = \prod_{i=1,4} (\lambda - \lambda_i), \quad (28)$$

where λ_i are zeros of the polynomial which characterize the periodic solution of the “Bloch wave” problem and are called the main spectra. The zeros λ_i have to consist of complex conjugate pairs $\lambda_j = \lambda_{Rj} + i\lambda_{Ij}$, $\lambda_{j+2} = \lambda_{Rj} - i\lambda_{Ij}$, $j = 1, 2$, which is required for that the obtained curve \mathbf{r} is real. Eqs. (27) and (28) are consistent with the following polynomial form of f, g and h in λ ,

$$f = \alpha_3 \lambda^2 + f_1 \lambda + f_0, \quad g = -i\lambda \alpha_+ (\lambda - \mu), \quad h = -i\lambda \alpha_- (\lambda - \mu^*), \quad (29)$$

where μ are functions of z, \bar{z} .

3.2 Equations for $\mu, f_i, \alpha_i, \beta_i, \gamma_i$

Using Eqs. (29) and (27), we can evaluate ∂g at $\lambda = \mu$ to find

$$\partial \mu = -i(\alpha_3 \mu^2 + f_1 \mu + f_0). \quad (30)$$

If we look at $O(\lambda^i), i = 2, 1, 0$ terms for ∂f in Eq. (27), we find

$$\partial \alpha_3 = -\frac{i}{2} \alpha_+ \alpha_- (\mu - \mu^*), \quad \partial f_1 = \partial f_0 = 0. \quad (31)$$

At $O(\lambda^2)$ terms for ∂g , we find

$$\partial \alpha_+ = i\alpha_3 \alpha_+ \mu + i\alpha_+ f_1. \quad (32)$$

Applying the same procedure to $\bar{\partial} g$ and $\bar{\partial} f$ in Eq. (27), we can obtain that

$$\begin{aligned} i\mu \alpha_+ \bar{\partial} \mu &= (\mu^3 \alpha_+ + 3\tau \mu^2 \alpha_+ + \frac{i}{2} \mu^2 \gamma_+ + \mu \beta_+) (\alpha_3 \mu^2 + f_1 \mu + f_0) \\ \bar{\partial} \alpha_+ &= i\beta_3 \alpha_+ \mu - \frac{1}{2} \gamma_+ f_0 + i\beta_+ f_1 + 3i\tau \alpha_+ f_0, \\ 0 &= \alpha_3 \alpha_+ \mu + \alpha_+ f_1 + \frac{i}{2} (\gamma_+ \alpha_3 - \gamma_3 \alpha_+), \\ 0 &= \frac{i}{2} \gamma_3 \alpha_+ \mu - \beta_3 \alpha_+ + 3\tau \alpha_3 \alpha_+ \mu + \frac{i}{2} \gamma_+ f_1 + \alpha_+ f_0 + \beta_+ \alpha_3 + 3\tau \alpha_+ f_1, \end{aligned} \quad (33)$$

and

$$\begin{aligned} \bar{\partial} \alpha_3 &= \frac{i}{2} (\beta_+ \alpha_- \mu^* - \beta_- \alpha_+ \mu), \quad \bar{\partial} f_1 = \bar{\partial} f_0 = 0, \\ 0 &= 2\alpha_+ \alpha_- (\mu^* - \mu) - i(\gamma_+ \alpha_- - \gamma_- \alpha_+), \\ 0 &= i(\gamma_+ \alpha_- \mu^* - \gamma_- \alpha_+ \mu) - 2(\beta_+ \alpha_- - \beta_- \alpha_+) + 6\tau \alpha_+ \alpha_- (\mu^* - \mu). \end{aligned} \quad (34)$$

3.3 $\mu, f_i, \gamma_i, \beta_i$ in terms of α_i

To prevent the “effectivization” problem, we start with Eqs. (28) and (29) to solve μ, f_1 and f_0 in terms of α_3 . First, we introduce constants of motion s_i , $i = 1, 4$ which are defined

$$P(\lambda) = f^2 - gh = \lambda^4 - s_1\lambda^3 + s_2\lambda^2 - s_3\lambda + s_4. \quad (35)$$

By inserting f, g and h in Eq. (29) into Eq. (35), we can obtain

$$\begin{aligned} s_1 &\equiv \sum \lambda_i = -2f_1\alpha_3 + \alpha_+\alpha_-(\mu + \mu^*), \\ s_2 &\equiv \sum_{i<j} \lambda_i\lambda_j = 2f_0\alpha_3 + f_1^2 + \alpha_+\alpha_-\mu\mu^*, \\ s_3 &\equiv \sum_{i<j<k} \lambda_i\lambda_j\lambda_k = -2f_1f_0, \quad s_4 \equiv \prod \lambda_i = f_0^2. \end{aligned} \quad (36)$$

Now solving for f_0, f_1, μ and μ^* in Eq. (36), we obtain

$$f_1 = \frac{s_3}{2\sqrt{s_4}}, \quad f_0 = -\sqrt{s_4}, \quad \mu, \mu^* = \frac{1}{2(1 - \alpha_3^2)s_4} \left(\sqrt{s_4}s_3\alpha_3 + s_1s_4 \mp \sqrt{s_4R} \right), \quad (37)$$

where

$$\begin{aligned} R &= (x + \lambda_2\lambda_3 + \lambda_1\lambda_4)(x + \lambda_1\lambda_3 + \lambda_2\lambda_4)(x + \lambda_1\lambda_2 + \lambda_3\lambda_4), \\ x &= 2\sqrt{s_4}\alpha_3. \end{aligned} \quad (38)$$

Note that $f_i, i = 0, 1$ are constants, which are consistent with Eqs. (31) and (34). To obtain this solution, we use Mathematica for symbolic manipulation. In addition, Mathematica was used to check various formulae appeared in this paper. For example, it was used to check that Eqs. (55) and (64) satisfies the equation of motion (23).

The definition of $\gamma_i, i = 1, 3$ in Eq. (24) together with Eqs. (31), (32) and (36) give

$$\begin{aligned} \gamma_3 &= -i\alpha_+\alpha_-(\mu + \mu^* + 2f_1/\alpha_3)\alpha_3 = -i(2f_1/\alpha_3 + s_1)\alpha_3 \equiv t\alpha_3, \\ \gamma_+ &= \{2i(\mu + \alpha_3f_1) - i\alpha_+\alpha_-(\mu + \mu^*)\}\alpha_+ = i(2\mu - s_1)\alpha_+ \equiv y\alpha_+, \end{aligned} \quad (39)$$

and $\gamma_- = -y^*\alpha_-$. Note that these results are consistent with the third equations of Eqs. (33) and (34).

Now, we require that α_3 becomes a traveling wave in the 1-phase solution, i.e., a function of $w \equiv z + v\bar{z}$, where v is a constant, requiring $\bar{\partial}\alpha_3 = v\partial\alpha_3$. The definition of α_3, β_3 in Eq. (24) gives $\bar{\partial}\alpha_3 = \partial\beta_3$, which in turn becomes

$$\beta_3 = v\alpha_3 + c(\bar{z}). \quad (40)$$

To fix $c(\bar{z})$, we need following steps. The fourth equations of Eqs. (33) and (34) together with the results in Eqs. (39), (40) give

$$\beta_+ = \{f_1^2/\alpha_3^2 + \mu f_1/\alpha_3 - f_0/\alpha_3 + v + c(\bar{z})/\alpha_3 - \frac{i}{2}(f_1/\alpha_3 + \mu)t - 3\tau(f_1/\alpha_3 + \mu)\}\alpha_+ \equiv u\alpha_+, \quad (41)$$

and $\beta_- = u^*\alpha_-$. Now, the first equation of Eq. (34) becomes

$$\bar{\partial}\alpha_3 = \frac{i}{2}\alpha_+\alpha_-(u\mu^* - u^*\mu). \quad (42)$$

Now, the relation $\bar{\partial}\alpha_3 = v\partial\alpha_3$ with Eqs. (31) and (41) gives $c(\bar{z}) = c$ is a constant and

$$c = f_0 + f_1 s_1/2 + 3\tau f_1. \quad (43)$$

Finally, inserting all these results into the first equation of Eq. (33), we can obtain

$$\bar{\partial}\mu = -i(\mu^2 + 3\tau\mu + \frac{i}{2}\mu y + u)(\alpha_3\mu^2 + f_1\mu + f_0) = v\partial\mu, \quad (44)$$

which shows μ is also a function of $w = z + v\bar{z}$ only.

3.4 Imposing the conditions $W = 3\tau^2 - \kappa^2/2$ and $\tau = \text{constant}$

To find the value of v , we need to impose the condition $W = 3\tau^2 - \kappa^2/2$, i.e., $\text{Tr}\partial r\bar{\partial}r = 6\tau^2 - \kappa^2$. First, we calculate the curvature κ . Using Eqs. (11), (24), (31), (32), (36), we find

$$2\kappa^2 = \text{Tr}(\partial^2 r)^2 = -\frac{1}{2}s_1^2 + 2s_2 - 4\alpha_3 f_0. \quad (45)$$

Similarly, we can obtain

$$\text{Tr}\partial r\bar{\partial}r = 2v + 2\alpha_3 f_0 - s_1^2/2, \quad (46)$$

which together with Eq. (45) gives

$$v = \frac{3}{8}s_1^2 - \frac{1}{2}s_2 + 3\tau^2 + \frac{3}{2}\tau s_1. \quad (47)$$

Now, we calculate the torsion and impose the condition that the torsion is a constant. We first find $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ vectors as following,

$$\begin{aligned} t_i \sigma_i &= \partial r = \begin{pmatrix} \alpha_3 & \alpha_+ \\ \alpha_- & -\alpha_3 \end{pmatrix}, \\ n_i \sigma_i &= \frac{1}{\kappa} \partial t_i \sigma_i = \frac{i}{\kappa} \begin{pmatrix} \alpha_- \alpha_+ (\mu^* - \mu)/2 & \alpha_+ (\alpha_3 \mu + f_1) \\ -\alpha_- (\alpha_3 \mu^* + f_1) & -\alpha_- \alpha_+ (\mu^* - \mu)/2 \end{pmatrix}, \\ b_i \sigma_i &= (\mathbf{n} \times \mathbf{t})_i \sigma_i = \frac{1}{\kappa} \begin{pmatrix} \alpha_3 s_1/2 + f_1 & \alpha_+ (s_1/2 - \mu) \\ \alpha_- (s_1/2 - \mu^*) & -\alpha_3 s_1/2 - f_1 \end{pmatrix}. \end{aligned} \quad (48)$$

Inserting the result in Eq. (48) into the equation $\partial b_3 = -\tau n_3$ gives (use Eqs. (45) and (31))

$$\left(\frac{s_1}{2} + \tau\right)\kappa^2 + f_0\left(\frac{s_1}{2}\alpha_3 + f_1\right) = 0, \quad (49)$$

which can be satisfied with a constant τ by choosing (use Eq. (45))

$$s_1 = -4\tau, \quad s_3 = 2\tau(4\tau^2 - s_2). \quad (50)$$

We check that the curvature κ given by Eq. (45) becomes the solution of the mKdV equation when s_1, s_3 satisfies the conditions in Eq. (50). Using Eq. (36), the condition in Eq. (50) can be expressed in terms of $\lambda_{R1}, \lambda_{R2}, \lambda_{I1}, \lambda_{I2}$ where $\lambda_j = \lambda_{Rj} + i\lambda_{Ij}$, $\lambda_{j+2} = \lambda_{Rj} - i\lambda_{Ij}$, $j = 1, 2$.

type-A : $\lambda_{R1} = \lambda_{R2} = -\tau$, λ_{I1} and λ_{I2} are arbitrary,

type-B : $\lambda_{R1} = -\lambda_{R2} - 2\tau$, $\lambda_{I1} = \lambda_{I2}$.

Thus there exist two types of curves of the mKdV equation, which require different choice of the main spectra λ_i . Each type of main spectra has two independent parameters for curves of torsion τ . They are λ_{I1} and λ_{I2} for the Type-A, while they are λ_{R1} and λ_{I1} for type-B. In addition, we require the condition on v in Eq. (47).

3.5 Derivation of r_3

Using all these results, we can obtain the equation for α_3 ,

$$\frac{d\alpha_3}{dw} = \frac{i}{2\sqrt{s_4}}\sqrt{R}. \quad (51)$$

This equation can be integrated in terms of Weierstrass' $\mathcal{P}(u, g_2, g_3)$ function. As far as Weierstrass elliptic functions are involved, we employ terminology and notation of [26] without further explanations. Explicitly

$$\alpha_3 = \frac{1}{2\sqrt{s_4}} \left(-\frac{s_2}{3} - 4\mathcal{P}(w + w_3, g_2, g_3) \right), \quad (52)$$

where w_3 is an integration constant and

$$\begin{aligned} g_2 &= \frac{1}{12}s_2^2 - \frac{1}{4}s_1s_3 + s_4, \\ g_3 &= \frac{1}{216}s_2^3 - \frac{1}{48}s_1s_2s_3 + \frac{1}{16}s_3^2 - \frac{1}{6}s_2s_4 + \frac{1}{16}s_1^2s_4. \end{aligned} \quad (53)$$

The integration constant w_3 is determined by the initial condition, which we shall choose as follows; $\mathcal{P}(w_3) = e_3$ at $w = 0$, where e_3 is the smallest root of the equation $4z^3 - g_2z - g_3 = 0$. This condition guarantees that $|\alpha_3| \leq 1$. Other two roots are denoted by e_1 and e_2 with $e_1 > e_2 > e_3$. w_3 as well as w_1 are called the half period of the \mathcal{P} function. They satisfy $\mathcal{P}(w_1) = e_1, \mathcal{P}(w_2) = e_2, e_1 + e_2 + e_3 = w_1 + w_2 + w_3 = 0$. Especially,

$$e_1 = \frac{1}{4}(\lambda_1\lambda_3 + \lambda_2\lambda_4) - \frac{s_2}{12}, \quad e_2 = \frac{1}{4}(\lambda_1\lambda_4 + \lambda_2\lambda_3) - \frac{s_2}{12}, \quad e_3 = \frac{1}{4}(\lambda_1\lambda_2 + \lambda_3\lambda_4) - \frac{s_2}{12}. \quad (54)$$

Now using $\alpha_3 = \partial r_3$ and $\beta_3 = \bar{\partial} r_3$ in Eq. (40), we can obtain (use Eq. (43))

$$\begin{aligned} r_3 &= \int \alpha_3 dw + \frac{s_3 s_1 - 4s_4 + 6\tau s_3}{4\sqrt{s_4}} \bar{z} \\ &= \frac{2}{\sqrt{s_4}} \left\{ \zeta(w + w_3, g_2, g_3) - \frac{s_2}{12} w + \frac{1}{8}(s_3 s_1 - 4s_4 + 6\tau s_3) \bar{z} - \eta_3 \right\}, \end{aligned} \quad (55)$$

where $\zeta(u, g_2, g_3)$ is the Weierstrass' zeta function and the integration constant η_3 is taken to be $\zeta(w_3, g_2, g_3)$. (We will use the notation $\eta_i = \zeta(w_i, g_2, g_3), i = 1, 3$ in the following.)

3.6 Derivation of r_+

Now we try to obtain the solution $r_+ = r_1 - ir_2$ of the mKdV-type curve. The procedure is similar as in [20, 21], and we will only describe some important steps of derivations. More details including the identities of Weierstrass' functions needed in the derivation can be found in [20, 21]. Using Eqs. (39), (40), (41), we can express the second equation of Eq. (33) as

$$\bar{\partial} \alpha_+ = v \partial \alpha_+ + i(s_1/2 + 3\tau) f_0 \alpha_+, \quad (56)$$

which, with the help of Eq. (32), gives

$$\alpha_+ = \exp\{-i\sqrt{s_4}(s_1/2 + 3\tau)\bar{z}\} \tilde{\alpha}_+(w), \quad (57)$$

where $\tilde{\alpha}_+$ satisfies the following differential equation,

$$\frac{d\tilde{\alpha}_+}{dw} = i\tilde{\alpha}_+(f_1 + \alpha_3\mu). \quad (58)$$

Using Eqs. (37), (51), (52), we can integrate Eq. (58) to obtain (see more details in [20, 21])

$$\tilde{\alpha}_+ = 2 \frac{i}{\sqrt{s_4}} \frac{\sigma(w + w_3 + \kappa_1) \sigma(w + w_3 + \kappa_2)}{\sigma(\kappa_1) \sigma(\kappa_2) \sigma^2(w + w_3)} \exp\{-\zeta(\kappa_1)w - \zeta(\kappa_2)w + \delta\}, \quad (59)$$

where two constants κ_1, κ_2 are defined by the following relations (Please distinguish the constants κ_1 and κ_2 from the curvature κ .),

$$\mathcal{P}(\kappa_1), \mathcal{P}(\kappa_2) = -s_2/12 \pm \sqrt{s_4}/2, \quad (60)$$

$$\mathcal{P}'(\kappa_1), \mathcal{P}'(\kappa_2) = -\frac{\sqrt{s_4}}{2} \frac{d\alpha_3}{dw} \Big|_{\alpha_3=\mp 1} = -\frac{i}{4} \sqrt{R} \Big|_{\alpha_3=\mp 1} = -\frac{i}{4} (s_1 \sqrt{s_4} \mp s_3). \quad (61)$$

The integration constant δ can be fixed by requiring $\alpha_3^2 + |\alpha_+|^2 = 1$. Explicit calculation of this requirement, especially at $w = 0$, gives $\delta = -\eta_3(\kappa_1 + \kappa_2)$.

We now derive r_+ . Using $\partial r_+ = \alpha_+$ in Eq. (24), we obtain

$$r_+ = \int \alpha_+ dz + M(\bar{z}) = \exp\{-i\sqrt{s_4}(s_1/2 + 3\tau)\bar{z}\} \int \tilde{\alpha}_+ dw + M(\bar{z}), \quad (62)$$

where $M(\bar{z})$ is a function to be determined. To evaluate the integration explicitly as well as to determine $M(\bar{z})$, we substitute Eq. (62) into the following relation,

$$\bar{\partial} r_+ = \beta_+ = u\alpha_+ = (v - s_1\mu/2 - 3\tau\mu)\alpha_+. \quad (63)$$

It then gives $M(\bar{z}) = 0$ and (see more details in [20, 21])

$$\begin{aligned} r_+ &= \frac{2i}{s_4} \{ \zeta(w + w_3 + \kappa_1) - \zeta(w + w_3 + \kappa_2) - \zeta(\kappa_1) + \zeta(\kappa_2) \} \\ &\times \frac{\sigma(w + w_3 + \kappa_1)\sigma(w + w_3 + \kappa_2)}{\sigma(\kappa_1)\sigma(\kappa_2)\sigma^2(w + w_3)} \\ &\times \exp\{-\zeta(\kappa_1)w - \zeta(\kappa_2)w - \eta_3(\kappa_1 + \kappa_2) - i\sqrt{s_4}(s_1/2 + 3\tau)\bar{z}\}. \end{aligned} \quad (64)$$

Eqs. (55), (64) are the main results of the present paper. The space curves of the mKdV equation are described by taking two types of the main spectra $\lambda_i, i = 1, 4$ shown in section 3.4. They are curves which have a constant torsion τ and their curvature κ is given by Eqs. (45) and (52).

4 Special Cases

In this section, we study some special cases of the obtained solution by taking specific values on the main spectra λ_i . The Weierstrass functions reduce to simple forms (for example, sinusoidal or elliptic functions) on these special limits. These cases contain the straight line, the Kelvin wave and the 1-solitonic curve, as well as plane curves in terms of Jacobi functions, and the closed ring.

4.1 The straight line

The simplest solution that corresponds to the straight line is obtained by taking $\lambda_{I1} = \lambda_{I2} = 0$ in the type-A curve (or $\lambda_{R1} = -\tau, \lambda_{I1} = 0$ in the type-B curve), i.e., $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = -\tau$ (real). In this case, $g_2 = g_3 = e_3 = 0$. The Weierstrass functions in this limit are given by $\mathcal{P}(u, 0, 0) = 1/u^2$, $\zeta(u, 0, 0) = 1/u$, $\sigma(u, 0, 0) = u$. And $w_3 = \kappa_1 = \infty$. Using these relations, we can easily obtain $r_3 = -z + 3\tau^2\bar{z}, r_+ = r_1 - ir_2 = 0$. The rotational and translational symmetry of the vortex equation can give more general configuration of the straight line [20].

4.2 The Kelvin wave

The next example, known as the Kelvin wave in fluid mechanics [27], corresponds to taking $\lambda_{I1} = 0$ in the type-A curve, i.e., $\lambda_1 = \lambda_3 = -\tau, \lambda_2 = -\tau + i\gamma, \lambda_4 = -\tau - i\gamma$. In this case, $g_2 = \frac{1}{12}\gamma^4, g_3 = \frac{1}{216}\gamma^6$. As $\Delta \equiv g_2^3 - 27g_3^2 = 0$, the Weierstrass functions are given in simple forms,

$$\begin{aligned}\mathcal{P}(u) &= -\gamma^2/12 + \gamma^2 \csc^2(\gamma u/2)/4, \\ \zeta(u) &= \gamma^2 u/12 + \gamma \cot(\gamma u/2)/2, \\ \sigma(u) &= 2 \exp(\gamma^2 u^2/24) \sin(\gamma u/2)/\gamma.\end{aligned}\tag{65}$$

And $e_2 = e_3 = -\gamma^2/12$, $w_3 = i\infty$,

$$\sin(\gamma\kappa_1/2), \sin(\gamma\kappa_2/2) = \frac{i\gamma}{\sqrt{2\tau^2 \pm 2\tau\sqrt{\tau^2 + \gamma^2}}},\tag{66}$$

$$\cot(\gamma\kappa_1/2), \cot(\gamma\kappa_2/2) = -\frac{i}{\gamma} \left(\sqrt{\tau^2 + \gamma^2} \pm \tau \right).\tag{67}$$

Now, a straightforward calculation gives

$$\begin{aligned}r_3 &= \frac{1}{\sqrt{\tau^2 + \gamma^2}} \{ \tau z + 3(\tau\gamma^2/2 + \tau^3)\bar{z} \}, \\ r_+ &= i \frac{\gamma}{\tau^2 + \gamma^2} \exp\{i\sqrt{\tau^2 + \gamma^2}(z + \tau^2\bar{z} - \gamma^2\bar{z}/2)\}.\end{aligned}\tag{68}$$

This curve is a helix ($r_+ = r_1 - ir_2$). When $\tau = 0$, it describes a circular ring having a constant radius and rotating on the xy -plane. The parameter γ is the curvature κ of the curve, i.e., $\kappa = \gamma$, while τ is the torsion of the curve. The three orthogonal vectors are given

by

$$\begin{aligned}
\mathbf{t} &= \frac{1}{\sqrt{\tau^2 + \kappa^2}}(-\kappa \cos \Theta, \kappa \sin \Theta, \tau), \\
\mathbf{n} &= (\sin \Theta, \cos \Theta, 0), \\
\mathbf{b} &= \frac{1}{\sqrt{\tau^2 + \kappa^2}}(\tau \cos \Theta, -\tau \sin \Theta, \kappa),
\end{aligned} \tag{69}$$

where $\Theta = \sqrt{\tau^2 + \kappa^2}(z + \tau^2 \bar{z} - \kappa^2 \bar{z}/2)$.

4.3 The Hasimoto 1-solitonic curve

To obtain the curves of 1-soliton from our periodic solution, we take $\lambda_{I1} = \lambda_{I2}$ in the type-A curve (or $\lambda_{R1} = -\tau$ in the type-B curve), i.e., $\lambda_1 = \lambda_2 = -\tau + i\gamma$, $\lambda_3 = \lambda_4 = -\tau - i\gamma$. In this case $g_2 = \frac{4}{3}\gamma^4$, $g_3 = -\frac{8}{27}\gamma^6$ and $w_3 = i\pi/(2\gamma)$, $e_3 = \mathcal{P}(w_3) = -2\gamma^2/3$, $\eta_3 = \zeta(w_3) = -i\gamma\pi/6$, $\sigma(w_3) = i \exp(\pi^2/24)/\gamma$. The Weierstrass functions are given by

$$\begin{aligned}
\mathcal{P}(w) &= -2\gamma^2/3 + \gamma^2 \coth^2(\gamma w), \\
\zeta(w) &= -\gamma^2 w/3 + \gamma \coth(\gamma w), \\
\sigma(w) &= \exp(-\gamma^2 w^2/6) \sinh(\gamma w)/\gamma.
\end{aligned} \tag{70}$$

And $\kappa_1 = \infty$, $\sinh \gamma \kappa_2 = -i\gamma/\sqrt{\tau^2 + \gamma^2}$, $\coth \gamma \kappa_2 = i\tau/\gamma$. It then gives

$$\begin{aligned}
r_3 &= -z - 3\tau^2 \bar{z} + \frac{2\gamma}{\tau^2 + \gamma^2} \tanh(\gamma z - \gamma^3 \bar{z}), \\
r_+ &= -\frac{2i\gamma}{\tau^2 + \gamma^2} \text{sech}(\gamma z - \gamma^3 \bar{z}) \exp(-i\tau z - i\tau^3 \bar{z}).
\end{aligned} \tag{71}$$

The curvature of the curve is $\kappa = 2\gamma \text{sech}(\gamma z - \gamma^3 \bar{z})$. It satisfies the mKdV equation in Eq. (7). The three orthogonal vectors can be easily constructed. For example,

$$\begin{aligned}
\mathbf{t} &= \frac{2\gamma \text{sech} \Omega}{\tau^2 + \gamma^2}(\gamma \sin \Theta \tanh \Omega - \tau \cos \Theta, -\gamma \cos \Theta \tanh \Omega - \tau \sin \Theta, \\
&\quad \gamma \text{sech} \Omega - \frac{\tau^2 + \gamma^2}{2\gamma} \cosh \Omega),
\end{aligned} \tag{72}$$

where $\Omega = \gamma z - \gamma^3 \bar{z}$, $\Theta = \tau z + \tau^3 \bar{z}$. In Figure 1, we show an example of the 1-solitonic curve with parameters $\gamma = 1/3$ and (a) $\tau = 1$, (b) $\tau = 3$.

4.4 The rigid vortex filament

When we take $\lambda_{I2} = \lambda_{R2} + \tau$ in the type-B curve, i.e., $\lambda_1 = \lambda_3^* = -a - 2\tau + i(a + \tau)$, $\lambda_2 = \lambda_4^* = a + i(a + \tau)$ for arbitrary a , then $v = 0$ (use Eq. (47)) and $w = (z - v\bar{z}) \rightarrow z$. Thus

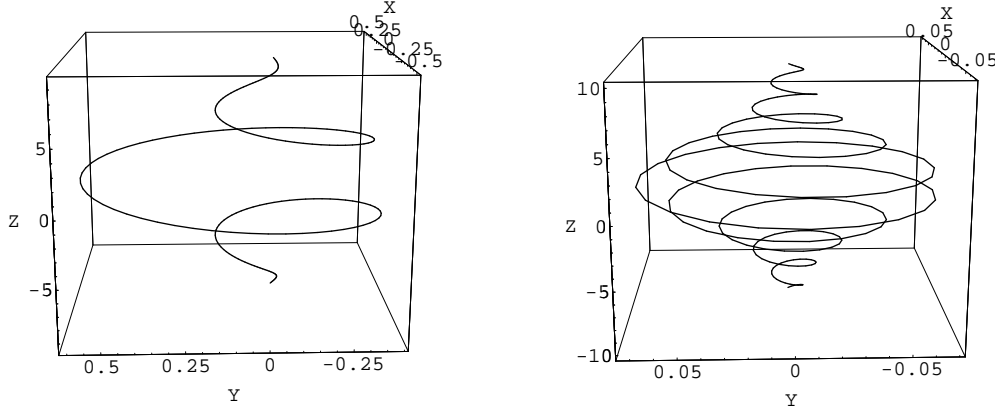


Figure 1: Typical 1-soliton curves drawn for $-10 < z < 10$, (a) with $\tau = 1$, (b) with $\tau = 3$. The parameter of the curves is $\gamma = 1/3$, drawn at $\bar{z} = 0$.

it describes a rigid filament of fixed shape, which rotates around the z -axis with constant angular velocity $(= -\tau\sqrt{4(a+\tau)^4 + \tau^4})$ and moves along the z -axis with constant velocity $(= -(4(a+\tau)^4 + 3\tau^4)/\sqrt{4(a+\tau)^4 + \tau^4})$. This configuration could be easily identified in experiments.

4.5 Plane curves in terms of Jacobi functions

Another reduced form of our solution is obtained by choosing $\kappa_1 = w_1, \kappa_2 = w_3$. Then $\mathcal{P}(\kappa_1) = \mathcal{P}(w_1) = e_1, \mathcal{P}(\kappa_2) = \mathcal{P}(w_3) = e_3$, and Eqs. (60) and (54) fix the main spectra as $\lambda_1 = \lambda_3^* = \rho \exp(i\theta), \lambda_2 = \lambda_4^* = -\rho \exp(-i\theta)$. Note that it corresponds to taking $\tau = 0$ in the type-B curve, i.e., $\lambda_{R1} = \rho \cos \theta, \lambda_{I1} = \rho \sin \theta$. The Weierstrass σ function is reduced to the Jacobi's elliptic function under this limit as following,

$$\frac{\sigma(w + w_3 + \kappa_1)\sigma(w + w_3 + \kappa_2)}{\sigma^2(w + w_3)} = \exp\left(\frac{\eta_1}{2w_1}(w_1^2 + w_3^2 - 2w_2w_3 - 2w_2w)\right) \times q^{-3/4} \exp\left(-i\frac{\pi}{2w_1}w\right) \sqrt{\frac{k}{k'}} \text{sn}(\sqrt{e_1 - e_3}w) \text{dn}(\sqrt{e_1 - e_3}w). \quad (73)$$

Using the relation $\eta_1 w_2 = -i\pi/2 + \eta_2 w_1$ (Legendre's relation) and $k = \sqrt{e_2 - e_3/e_1 - e_3} = \sin \theta, k' = \cos \theta, q = \exp(i\pi w_3/w_1)$, we can obtain

$$\begin{aligned} r_+ &= 2i \sin \theta \text{cn}(\rho w)/\rho, \\ r_3 &= \int (-1 + 2\text{dn}^2(\rho w)) dw - \rho^2 \bar{z} = -w + 2E(\rho w|k)/\rho - \rho^2 \bar{z}, \end{aligned} \quad (74)$$

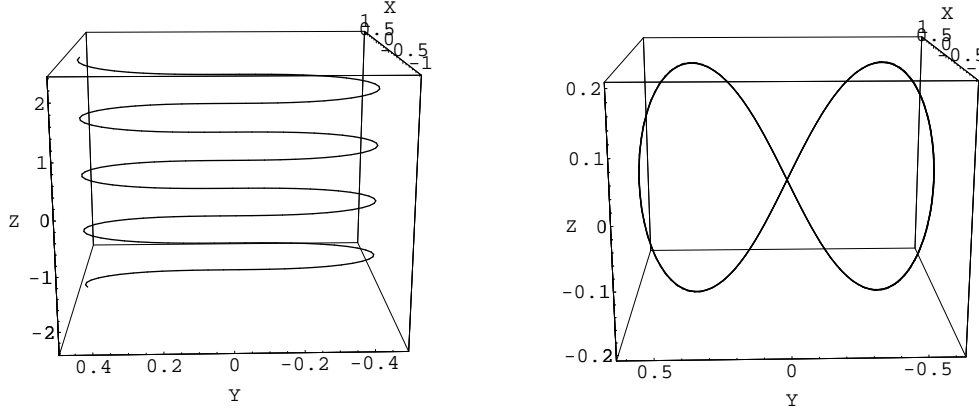


Figure 2: (a) Open plane curve for $-10 < z < 10$ and $\bar{z} = 0$. (b) Closed plane curve showing the famous figure-8 shape.

where E is the incomplete elliptic integral of the second kind. The curve lies on the yz -plane, and has the curvature $\kappa = 2k\rho\text{cn}\{z - \rho^2(2k^2 - 1)\bar{z}\}$ and torsion $\tau = 0$. Figure 2 show examples of the plane curve. Fig. 2(a) shows an open plane curve which is obtained by taking $\lambda_1 = \lambda_3^* = (3 + 3i)/\sqrt{2}$, $\lambda_2 = \lambda_4^* = (-3 + 3i)/\sqrt{2}$, $\tau = 0$. Fig. 2(b) shows the famous figure-8 shape (closed plane curve) in [4, 10], which is obtained by taking special value $\theta = 1.14$. The condition for closed curves will be discussed in the following section.

4.6 Closed curves

The characteristics of closed curves can be obtained from the following quasi-periodicity properties of Weierstrass' elliptic functions;

$$\begin{aligned}\zeta(u + 2w_i) &= \zeta(u) + 2\eta_i, \\ \sigma(u + 2w_i) &= -\exp\{2\eta_i(u + w_i)\}\sigma(u).\end{aligned}\tag{75}$$

In our problem, w_1 is real, while w_3 is pure imaginary. Thus the physical characteristics of closed curves are described by the period $\Delta w = 2w_1$. After this period $2w_1$, r_+ obtains an additional factor

$$\exp\{2\eta_1(\kappa_1 + \kappa_2) - 2\zeta(\kappa_1)w_1 - 2\zeta(\kappa_2)w_1\}.\tag{76}$$

Thus a necessary condition for a closed curve, i.e., $r_+(z = 2mw_1) = r_+(z = 0)$, is

$$\eta_1(\kappa_1 + \kappa_2) - w_1\{\zeta(\kappa_1) + \zeta(\kappa_2)\} = i\frac{n}{m}\pi,\tag{77}$$

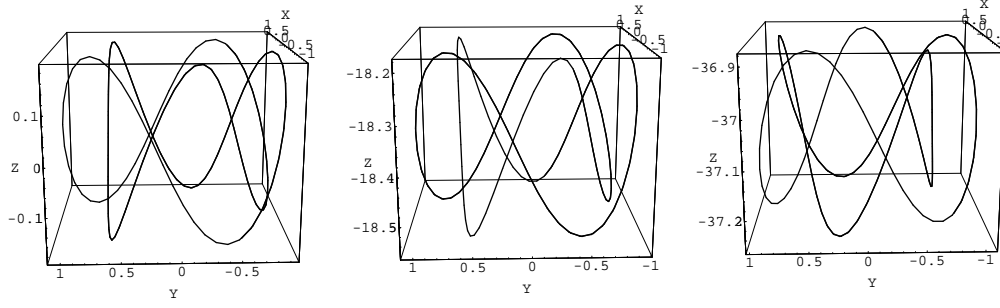


Figure 3: A typical closed ring plotted for $-10 < z < 10$ with parameters $\lambda_1 = \lambda_3^* = -2.0653 + 1.171i$, $\lambda_2 = \lambda_4^* = 0.0653 + 1.171i$, $\tau = 1$ (a) at $\bar{z} = 0$, (b) at $\bar{z} = 5$, (c) at $\bar{z} = 10.09$ (quasi-period time).

where m, n are arbitrary integers. Another condition comes from the closedness of the z -coordinate, i.e., $r_3(z = 2mw_1) = r_3(z = 0)$, which is

$$-\frac{s_2}{12\sqrt{s_4}}w_1 + \frac{1}{\sqrt{s_4}}\eta_1 = 0, \quad (78)$$

i.e., $s_2w_1 = 12\eta_1$.

The quasi-periodic property of the closed curves along the time is following. During a time period $\Delta\bar{z} = 2w_1/v$, the ring returns to its original shape, but rotates around the z -axis by an angle

$$2\frac{n}{m}\pi - \sqrt{s_4}(s_1 + 6\tau)\frac{w_1}{v}, \quad (79)$$

and moves a distance

$$\frac{s_3s_1 - 4s_4 + 6\tau s_3}{2\sqrt{s_4}}\frac{w_1}{v} \quad (80)$$

along the z -axis.

The two conditions, Eqs. (77) and (78), can be solved numerically using a software package like Mathematica. One possible solution is $\lambda_1 = \lambda_3^* = -2.0653 + 1.171i$, $\lambda_2 = \lambda_4^* = 0.0653 + 1.171i$, $\tau = 1$, which belongs to type-B curves. It satisfies the conditions (77) and (78) with $n = 2, m = 5$ ($w_1 = -1.198, v = -0.2374$). Figure 3 shows the motion of a closed curve at (a) $\bar{z} = 0$, (b) $\bar{z} = 5$, (c) $\bar{z} = 10.09$ ($=\Delta\bar{z}$, quasi-period time), respectively. During a time period $\Delta\bar{z}$, the ring rotates -25.6 radian and advances -37.07 along the z -axis. The z period is $10w_1 = -11.98$. We can't find the type-A curve that satisfies the closedness conditions in Eqs. (77) and (78).

In the case of figure-8 shape (closed plane curve), the condition $\kappa_1 = w_1, \kappa_2 = w_3$ satisfies Eq. (77) with $n = 1, m = 2$ (Legendre's relation.). The second condition (78) becomes

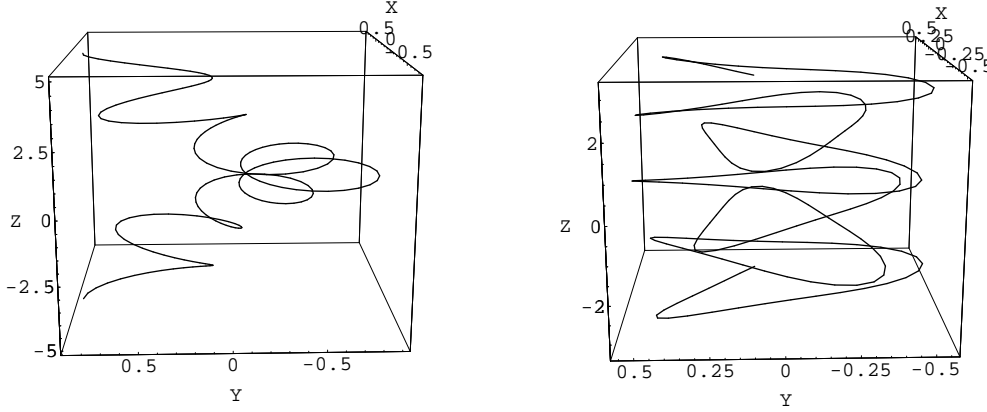


Figure 4: Typical open curves having $\tau = 1$, drawn for $-10 < z < 10$, $\bar{z} = 0$, (a) type-A curve with $\lambda_1 = \lambda_3^* = -1 + i$, $\lambda_2 = \lambda_4^* = -1 + 2i$, and (b) type-B curve with $\lambda_1 = \lambda_3^* = -3 + 2i$, $\lambda_2 = \lambda_4^* = 1 + 2i$.

$6\zeta(w_1) = -\rho^2 w_1$, which is satisfied by choosing $\lambda_1 = \lambda_3^* = \rho(1 + 2.177966i)$, $\lambda_2 = \lambda_4^* = -\rho(1 - 2.177966i)$, or $\theta = 1.14$ in the notation of section 4.5. Fig. 2(b) shows the figure-8 shape curve.

4.7 General vortex filament

Most generally, the solution in Eqs. (55) and (64) describes an open space curve with a constant torsion τ . Figure 4(a) shows an example (type-A curve), which we obtain using $\lambda_1 = \lambda_3^* = -1 + i$, $\lambda_2 = \lambda_4^* = -1 + 2i$, $\tau = 1$. Figure 4(b) shows another example (type-B curve), which we obtain using $\lambda_1 = \lambda_3^* = -3 + 2i$, $\lambda_2 = \lambda_4^* = 1 + 2i$, $\tau = 1$. These are examples of the most general configurations described by the 1-phase quasi-periodic solution.

5 Discussions

In this paper, we present the calculation of the one-phase quasi-periodic solution of space curves described by the mKdV equation. The solution is explicitly given in terms of Weierstrass' elliptic functions. The main difference of the present calculation compared to that of previous works in [19, 20, 21] is the appearance of constraints on the main spectra resulting from imposing two conditions. It then results in two types of curves, type-A and type-B curves. The first condition, $W = 3\tau^2 - \kappa^2/2$ in Section 2.4, is related with the internal parametrization of the curve. It does not determine the extrinsic form of the curve [10]. This is consistent with the fact that the curve equation in Equation (23) only fixes the form

of U and V and does not fix W in Equation. (2). As is shown in Section 3.4, the condition on W fixes the velocity v of the motion of rigid curve. The second condition, $\tau = \text{constant}$, is also unique in the present calculation, and is not required in the previous calculations in [19, 20, 21]. It was checked that the constraint on the main spectra is required for that κ in Equation (45) satisfies the mKdV equation in Equation (7).

We study various configurations resulting from the degenerate limit of the quasi-periodic solution. They contain the Kelvin wave belonging to the type-A curve, the rigid vortex filament, the Jacobian plane curve, and closed curves belong to the type-B curve, and the straight line and the one-solitonic curve belonging both to the type-A and type-B curve simultaneously.

The simple case of zero-torsion was studied in various contexts such as the elastica problem and the computer vision problem. In this case, the fact that the curve lies on a plane was the main keystone in the derivation of the curve configurations. In the case of non-zero torsion, the convenient formalism of differential geometry seems difficult to realize except in some especially simple cases. The present derivation heavily depends on the integrability structure of the mKdV equation. The equations for the one-phase solution in Section 3.2 are by no means simpler than the equations for the stationary curve configuration given in [11] (Equations (3.32), (3.36), (3.38), and (3.39)). But there exists a standard procedure in solving these equations, shown in Section 3.2. Moreover, this procedure offers a generalization which can be used in calculating the N -phase periodic solutions.

Our formalism might be adapted to the curve problem of the non-zero and non-constant torsion problem. It is known that the spectral parameter of the integrable equation can be extended to be z, \bar{z} -dependent without destroying the integrability [28, 29]. Thus it could result in space curves of non-constant torsion. Explicit construction of curves with non-constant torsion is rare and difficult. Thus it remains as an interesting work for the future.

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